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LETTER TO THE EDITOR

Continuous variable teleportation as a generalized thermalizing quantum channel

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Abstract

A quantum channel is derived for continuous variable teleportation which is performed by means of an arbitrary entangled state and the standard protocol. When a Gaussian entangled state such as a two-mode squeezed-vacuum state is used, the continuous variable teleportation is equivalent to the thermalizing quantum channel. Continuous variable quantum dense coding is also considered.

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Quantum information processing attracts much attention in quantum physics and information science [1–3], which gives a deeper insight into the principles of quantum mechanics and provides novel communication systems such as quantum teleportation and quantum dense coding. Quantum teleportation transmits an unknown quantum state by means of classical communication and quantum entanglement [4]. Quantum dense coding sends classical information via a quantum channel and quantum entanglement [5]. Bowen and Bose have recently shown that a finite-dimensional quantum teleportation with the standard protocol is equivalent to a generalized depolarizing quantum channel [6]. In this letter, we generalize this result to infinite-dimensional quantum teleportation and show that continuous variable quantum teleportation with the standard protocol is equivalent to a generalized thermalizing channel.

We explain the standard protocol of continuous variable quantum teleportation [7, 8]. Alice and Bob first share an entangled quantum system in a quantum state $\hat{W}^{AB}$ which is arbitrary though a two-mode squeezed-vacuum state is usually used, where A and B stand for the quantum system at Alice and Bob's sides. When Alice is given a quantum state $\hat{\rho}^{Q}$ to be teleported to Bob, she performs the simultaneous measurement of position and momentum of the compound quantum system $Q + A$. Then Alice sends the measurement outcome (x, p) to Bob via a classical communication channel. After receiving the measurement outcome, Bob applies the unitary transformation described by the displacement operator $\hat{D}(x, p) = \mathrm{e}^{\mathrm{i}(p\hat{x}-x\hat{p})}$

to the system B. Bob finally obtains the quantum state $\hat{\rho}^B$ which is, in an ideal case, identical to the quantum state $\hat{\rho}^Q$ to be teleported.

The main result of this letter is that the completely positive map $\hat{\mathcal{L}}$ which describes the continuous variable teleportation by means of an arbitrary entangled state and the standard protocol is given by

$$\hat{\mathcal{L}}\hat{\rho} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, \mathcal{P}(x, p)\hat{D}(x, p)\hat{\rho}\hat{D}^{\dagger}(x, p) \tag{1}$$

with

$$\mathcal{P}(x, p) = \langle\Phi^{AB}|[\hat{1}^A \otimes \hat{D}^B(x, p)]^{\dagger}\hat{W}^{AB}[\hat{1}^A \otimes \hat{D}^B(x, p)]|\Phi^{AB}\rangle \tag{2}$$

where $|\Phi^{AB}\rangle$ is a completely entangled state of continuous variable,

$$|\Phi^{AB}\rangle = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}x|x^A\rangle \otimes |x^B\rangle \tag{3}$$

which satisfies $\mathrm{Tr}_A|\Phi^{AB}\rangle\langle\Phi^{AB}| = (1/2\pi)\hat{1}^B$ and $\mathrm{Tr}_B|\Phi^{AB}\rangle\langle\Phi^{AB}| = (1/2\pi)\hat{1}^A$. In this state, the systems A and B have the same values of position and the opposite values of momentum, that is $(\hat{x}^A - \hat{x}^B)|\Phi^{AB}\rangle = (\hat{p}^A + \hat{p}^B)|\Phi^{AB}\rangle = 0$. When $\mathcal{P}(x, p) = P(x)P(p)$ and $P(x)$ is a Gaussian function of x, the completely positive map $\hat{\mathcal{L}}$ becomes the thermalizing quantum channel. Hence we refer to the completely positive map $\hat{\mathcal{L}}$ as a generalized thermalizing quantum channel. In a discrete quantum teleportation of an N-dimensional Hilbert space, this corresponds to a generalized depolarizing quantum channel obtained by Bowen and Bose [6]. When a pure state $\hat{\rho} = |\psi\rangle\langle\psi|$ is teleported, the fidelity F_{ψ} is calculated by the formula

$$F_{\psi} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p\, \mathcal{P}(x, p)|\langle\psi|\hat{D}(x, p)|\psi\rangle|^2. \tag{4}$$

The property of the simultaneous measurement of position and momentum is essential for deriving the completely positive map $\hat{\mathcal{L}}$. The simultaneous measurement which yields the measurement outcome (x, p) is described by a projection operator $\hat{\pi}^{AB}(x, p) = |\Phi^{AB}(x, p)\rangle\langle\Phi^{AB}(x, p)|$, where $|\Phi^{AB}(x, p)\rangle$ is given by

$$\begin{aligned} |\Phi^{AB}(x, p)\rangle &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \mathrm{d}y|x^A + y^A\rangle \otimes |y^B\rangle\, \mathrm{e}^{\mathrm{i}py} \\ &= [\hat{D}^A(x, p) \otimes \hat{1}^B]|\Phi^{AB}\rangle\, \mathrm{e}^{-\frac{1}{2}\mathrm{i}px} \\ &= [\hat{1}^A \otimes \hat{D}^B(-x, p)]|\Phi^{AB}\rangle\, \mathrm{e}^{-\frac{1}{2}\mathrm{i}px} \end{aligned} \tag{5}$$

with

$$\hat{D}(x, p) = \exp[\,\mathrm{i}(p\hat{x} - x\hat{p})] = \exp(\alpha\hat{a}^{\dagger} - \alpha\hat{a}) \tag{6}$$

and $\hat{a} = (\hat{x} + \mathrm{i}\hat{p})/\sqrt{2}$ and $\alpha = (x + \mathrm{i}p)/\sqrt{2}$. The state vector $|\Phi^{AB}(x, p)\rangle$ is the simultaneous eigenstate of position-difference operator $\hat{x}^A - \hat{x}^B$ and momentum-sum operator $\hat{p}^A + \hat{p}^B$, which satisfies the relations

$$\langle\Phi^{AB}(x, p)|\Phi^{AB}(x', p')\rangle = \delta(x - x')\delta(p - p') \tag{7}$$

$$\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p|\Phi^{AB}(x, p)\rangle\langle\Phi^{AB}(x, p)| = \hat{1}^A \otimes \hat{1}^B. \tag{8}$$

For quantum optical systems, the simultaneous measurement of position and momentum is implemented by a heterodyne detection [9].

Let us now derive the main result given by equations (1)–(3). Suppose that Alice and Bob share an arbitrary entangled state $\hat{W}^{AB}$ and Alice has a quantum state $\hat{\rho}^Q$ which is to be teleported to Bob. The total quantum state of Alice and Bob is given by $\hat{\rho}^Q \otimes \hat{W}^{AB}$. Alice performs the simultaneous measurement of position and momentum of the compound system $Q + A$, described by the projection operator $\hat{x}^{QA}(x, p)$, and informs Bob of the measurement outcome (x, p). Then Bob applies the unitary operator $\hat{D}^B(x, p)$ to the system B and finally obtains the quantum state

$$\hat{\rho}^B(x, p) = \frac{\hat{D}^B(x, p)\{\mathrm{Tr}_{QA}[(\hat{x}^{QA}(x, p) \otimes \hat{1}^B)(\hat{\rho}^Q \otimes \hat{W}^{AB})]\}\hat{D}^{B\dagger}(x, p)}{\mathrm{Tr}_{QAB}[(\hat{x}^{QA}(x, p) \otimes \hat{1}^B)(\hat{\rho}^Q \otimes \hat{W}^{AB})]} \tag{9}$$

with probability

$$P(x, p) = \mathrm{Tr}_{QAB}[(\hat{x}^{QA}(x, p) \otimes \hat{1}^B)(\hat{\rho}^Q \otimes \hat{W}^{AB})]. \tag{10}$$

In deriving equation (9), we have used the state-reduction formula [10–12]. Hence the teleported quantum state $\hat{\rho}^B_{\mathrm{out}}$ of Bob is given, in average, by

$$\hat{\rho}^B_{\mathrm{out}} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p \hat{D}^B(x, p)\{\mathrm{Tr}_{QA}[(\hat{x}^{QA}(x, p) \otimes \hat{1}^B)(\hat{\rho}^Q \otimes \hat{W}^{AB})]\}\hat{D}^{B\dagger}(x, p) \tag{11}$$

which determines the completely positive map $\hat{\mathcal{L}}$ of the continuous variable teleportation by means of an arbitrary entangled state and the standard protocol.

To obtain the completely positive map $\hat{\mathcal{L}}$ from equation (11), we expand the entangled quantum state $\hat{W}^{AB}$ in terms of $|\Phi^{AB}(x, p)\rangle$ as

$$\hat{W}^{AB} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}y \int_{-\infty}^{\infty} \mathrm{d}u \int_{-\infty}^{\infty} \mathrm{d}v F_W(x, u; y, v)|\Phi^{AB}(x, u)\rangle\langle\Phi^{AB}(y, v)| \tag{12}$$

where we set $F_W(x, u; y, v) = \langle\Phi^{AB}(x, u)|\hat{W}^{AB}|\Phi^{AB}(y, v)\rangle$. Note that the state vector $|\Phi^{AB}(x, p)\rangle$ satisfies the relation

$$\langle\Phi^{AB}(x, u)|\Phi^{BC}(y, v)\rangle = \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}z|z^C - y^C\rangle\langle z^A + x^A|\,\mathrm{e}^{-\mathrm{i}uz+\mathrm{i}v(z-y)}. \tag{13}$$

Substituting these equations into equation (11) and using $|x + y\rangle = \mathrm{e}^{-\mathrm{i}y\hat{p}}|x\rangle$, after some calculation, we obtain

$$\hat{\rho}^B_{\mathrm{out}} = \int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p F_W(x, p; x, p)\hat{D}^B(-x, p)\hat{\rho}^B\hat{D}^{B\dagger}(-x, p) \tag{14}$$

where $\hat{\rho}^B$ is the quantum state of the system B, which is identical to that described by $\hat{\rho}^Q$. Using equation (5), we can arrive at the result given by equations (1)–(3). Therefore, we have found that the continuous variable teleportation by means of an arbitrary entangled state and the standard protocol is equivalent to the generalized thermalizing quantum channel.

The continuous variable teleportation is usually performed by means of a two-mode squeezed-vacuum state $|\Psi^{AB}_{\mathrm{SV}}\rangle = \mathrm{e}^{r(\hat{a}^\dagger\hat{b}^\dagger - \hat{a}\hat{b})}|0^A, 0^B\rangle$ [7, 8]. In this case, calculating $\mathcal{P}(x, p) = |\langle\Psi^{AB}_{\mathrm{SV}}|\Phi^{AB}(x, p)\rangle|^2$ we obtain

$$\hat{\mathcal{L}}\hat{\rho} = \frac{1}{2\pi\bar{n}_r}\int_{-\infty}^{\infty} \mathrm{d}x \int_{-\infty}^{\infty} \mathrm{d}p \exp\left(-\frac{x^2 + p^2}{2\bar{n}_r}\right)\hat{D}(x, p)\hat{\rho}\hat{D}^\dagger(x, p) \tag{15}$$

where $\bar{n}_r = \mathrm{e}^{-2r}$ and r is the squeezing parameter of the two-mode squeezed-vacuum state $|\Psi^{AB}_{\mathrm{SV}}\rangle$. The completely positive map $\hat{\mathcal{L}}$ given by equation (15) is equivalent to the transfer-operator representation of the continuous variable teleportation [13]. When we denote the Glauber–Sudarshan P-function of the quantum state $\hat{\rho}$ as $P(\alpha)$, equation (15) can be rewritten as

$$\hat{\mathcal{L}}\hat{\rho} = \int \frac{\mathrm{d}^2\alpha}{\pi} P(\alpha)\hat{D}(\alpha)\hat{\rho}_{\mathrm{th}}\hat{D}^\dagger(\alpha) \tag{16}$$

where $\hat{\rho}_{\rm th}$ is the thermal state

$$\hat{\rho}_{\rm th} = \frac{1}{1+\bar{n}_r}\sum_{n=0}^{\infty}\left(\frac{\bar{n}_r}{1+\bar{n}_r}\right)^n |n\rangle\langle n|. \tag{17}$$

This result explicitly shows that the continuous variable teleportation by means of the two-mode squeezed-vacuum state is nothing but the thermalizing quantum channel. When the two-mode squeezed-vacuum state is shared through a noisy quantum channel, the parameter $\bar{n}_r = {\rm e}^{-2r}$ as appeared in equation (15) is replaced with $\bar{n}_r = 1-(1-{\rm e}^{-2r})T$ [14–16], where T stands for the transmission coefficient of the noisy quantum channel.

We next consider the continuous variable quantum dense coding by means of an arbitrary entangled state and the standard protocol in which Alice applies the unitary operator $\hat{D}(x,p)$ to encode some classical information and Bob performs the simultaneous measurement of position and momentum to extract the information [17–20]. Suppose that Alice and Bob share an arbitrary entangled quantum state $\hat{W}^{AB}$ to transmit classical information via the continuous variable quantum dense coding. When Alice applies the unitary operator $\hat{D}(x,p)$ to her system and sends it to Bob, his quantum state $\hat{W}^{AB}(x,p)$ is given by

$$\hat{W}^{AB}(x,p) = \int_{-\infty}^{\infty}{\rm d}y\int_{-\infty}^{\infty}{\rm d}z\int_{-\infty}^{\infty}{\rm d}u\int_{-\infty}^{\infty}{\rm d}v F_W(y,u;z,v) \times|\Phi^{AB}(x+y,p+u)\rangle\langle\Phi^{AB}(x+z,p+v)|. \tag{18}$$

Thus, the conditional probability (the channel matrix of the quantum dense coding) $P(x',p'|x,p)$ that Bob obtains the measurement outcome (x',p') when Alice encodes (x,p) is calculated to be

$$\begin{aligned}P(x',p'|x,p) &= \langle\Phi^{AB}(x'-x,p'-p)|\hat{W}^{AB}|\Phi^{AB}(x'-x,p'-p)\rangle\\ &= F_W(x'-x,p'-p;x'-x,p'-p)\\ &= \mathcal{P}(x-x',p'-p)\end{aligned} \tag{19}$$

where $\mathcal{P}(x,p)$ is given by equation (2). When $\hat{W}^{AB}$ is the two-mode squeezed-vacuum state, we obtain

$$P(x',p'|x,p) = \frac{1}{2\pi\bar{n}_r}\exp\left[-\frac{(x'-x)^2+(p'-p)^2}{2\bar{n}_r}\right]. \tag{20}$$

In the strong squeezing limit ($r \gg 1$), the channel matrix $P(x',p'|x,p)$ is approximated as $\delta(x'-x)\delta(p'-p)$ and thus the continuous variable quantum dense coding can transmit an amount of classical information twice that without quantum entanglement [18]. When the two-mode squeezed-vacuum state is sent through a noisy quantum channel, the parameter $\bar{n}_r = {\rm e}^{-2r}$ as appeared in equation (20) is replaced with $\bar{n}_r = 1-(1-{\rm e}^{-2r})T$ [19, 20].

In summary, we have shown that the continuous variable quantum teleportation by means of an arbitrary entangled quantum state and the standard protocol is equivalent to the generalized thermalizing quantum channel. This is a continuous version of the result obtained by Bowen and Bose [6], in which they have shown that the discrete (the finite-dimensional) quantum teleportation with the standard protocol is equivalent to the generalized depolarizing quantum channel. In particular, when a two-mode squeezed-vacuum state is used as an entanglement resource, the continuous variable quantum teleportation becomes the thermalizing channel in which the number of thermal photons is given by $\bar{n}_r = {\rm e}^{-2r}$. The continuous variable quantum dense coding by means of an arbitrary entangled state and the standard protocol has also been considered. As long as the standard protocol is applied, not only the continuous variable quantum teleportation but also the continuous variable quantum dense coding is completely determined by the function $\mathcal{P}(x,p)$ given by equation (2).

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